Unconditional Quantile Regression with Endogenous Regressors

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Abstract

This paper proposes an extension of the Fortin, Fipro and Lemieux (2009) unconditional quantile regression approach by allowing endogenous regressors in a nonseparable triangular model. The RIF (Re-centered Influence Function) regression has been identified by using the control variable approach developed by Imbens and Newey (2009). The results are not specific to models with endogenous regressors, but also applies to the exogenous case for which there is no need for a control variable. We also derive the Unconditional Quantile Partial Effect (UQPE) and RIF regression for the nonlinear additively separable model and show that Fipro et al. (2009) results can be obtained from our results by assuming no endogenous regressors in X. We have illustrated our theoretical results by using the Box-Cox unconditional quantile regression model as an example. We also show that in the nonlinear additively separable model, we can separate out the bias terms for UQPE and RIF regression functions by using the additive separability condition between regressors and error terms. We show that the bias terms for UQPE and RIF regression are non monotonic across the quantiles.

Keywords: Re-centered influence function, control variable, unconditional quantile partial effect, RIF regression, nonseparable model.

JEL Code: C14, C21, J31

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1
1 Introduction

The unconditional quantile regression approach has attracted lot of attention in the econometrics literature since Fipro, Fortin and Lemieux (2009) introduced the concept. One important factor behind this interest is that conditional quantiles do not average up to their unconditional population counterparts. As a result, the estimates obtained by running a quantile regression can not be used to estimate the impact of $X$ on the corresponding unconditional quantile. This implies that the quantile regression method cannot find out the impact of marginal increases in all the workers' education on some features of the distribution of wage, such as its moments, quantiles, Gini coefficient or other measure of inequality.

Fipro et al. (2009) introduce the RIF regression model in the presence of exogenous regressors only. The independence assumption between $X$ and the error terms ($\epsilon$) plays a very important role in their analysis. This assumption is very restrictive in most of the micro-economic applications, so this makes the unconditional quantile regression approach by Fipro et al. (2009) less feasible in many micro-economic applications. In this paper, one of the main goals is to relax the independence assumption and consistently estimate the unconditional quantile regression function.

Nonseparable economic models have received considerable popularity in the econometrics literature such as Chesher (2003), Matzkin (2003), Chesher (2005), Altonji and Matzkin (2005), Chernozhukov and Hansen (2005), Hoderlein and Mammen (2007), and Chernozhukov, Imbens and Newey (2007), amongst others. This is because this model does not put any restriction on the econometric specification between regressors and the disturbance terms. This way we allow individual heterogeneity in a fully flexible way. Another important feature of our model is that we include endogenous regressors, which is a central issue of many micro-economic models. Imbens and Newey (2009) provides identification and estimation results for this type of model by using the control variable approach.


Fipro et al. (2009) identify the Unconditional Partial Effects by showing that it can be presented by the average derivative of a projection of the re-centered influence function of the statistic of interest on the regressors under the condition that $X$ and $\epsilon$ are uncorrelated.
Rothe (2010) shows that this result can be generalized by the triangular nonseparable models as developed by Imbens and Newey (2009). In this paper, first we show a simpler version of Rothe’s (2010) results for the specific distributional statistic quantile whereas Rothe’s (2010) results hold for any distributional statistic. We reformulate the results for quantiles only because we want to identify the RIF regression function in the triangular nonseparable model.

In this paper, we also consider an additively separable model where covariates (X) and the error term (ε) are additively separable. We derive the UQPE and RIF regression and show Fipro et al.’s (2009) results can be obtained from our results if we assume the error term is uncorrelated with X. The results of this paper are not specific for the model with endogenous regressors but holds for models with exogenous regressors as well. We demonstrate our results by considering a Box-Cox unconditional quantile regression model as an example.

The structure of this paper is as follows: in the next section, we describe our theoretical model and discuss the identification method for the RIF regression in nonseparable and additively separable model, and I conclude in section 3. All the derivations are shown in the appendix.

2 Econometric Model

The model I consider in this paper is essentially the same as originally proposed by Imbens and Newey (2009) and then followed by Rothe (2010). The outcome variable (Y) depends on observable X and unobservable ϵ in the form:

\[ Y = h(X, \epsilon) \]  

(1)

Imbens and Newey (2009) point out that ε often represents individual heterogeneity and most likely correlated with X because X is an equilibrium outcome partially determined by ϵ. This is true for many micro-economic applications.

In a triangular system, the sub-vector \( X_1 \) of \( X \) is the set of endogenous variables and \( X_2 \) is the vector of exogenous variables so that \( X = (X_1', X_2')' \). There is a vector of instrument \( Z_1 \) and a scalar disturbance \( \eta \) such that \( Z = (Z_1', X_2') \). The reduced form for \( X_1 \) is given by,

\[ X_1 = h_1(Z, \eta) \]  

(2)

where \( h_1(Z, \eta) \) is strictly monotonic in \( \eta \). Imbens et al. (2009) do not put any restrictions on the dimensionality of \( \epsilon \). However, Rothe (2010) put some restrictions on dimensionality
for identification of the model.

An economic example helps to motivate this kind of model. The Mincer equation is the most widely used specification of the empirical earnings equation where the outcome variable is the individual’s income and dependent variables are observable human capital variables like education, experience and some other demographic variables. We note that the years of schooling depends on the unobserved characteristics such as ability, which become a part of the idiosyncratic error term $\epsilon$. This means $X$ and $\epsilon$ are correlated. Card (2001) and Das (2001) show another example by considering a similar model that corresponds to choices of education with heterogeneous returns.

Fipro, Fortin and Lemieux (2009) defined ‘Unconditional Partial Effects’ as the small location shift in the distribution of a continuous variable $X$ on the distributional statistic $\nu(F_Y)$,

$$UQPE(\nu) = \lim_{t \to 0} \frac{\nu(F_{Y,t}) - \nu(F_Y)}{t}$$

where $\nu(F_{Y,t}) = \int F_{Y|X}(y|x).dF_X(x-t)$ and $F_{Y|X}(\cdot)$ is a continuous and smooth function. Fipro, Fortin and Lemieux (2009) show that any distributional statistic $\nu(F_y)$ can be expressed in terms of the conditional expectation of the Re-centered Influence Function ($RIF(y; \nu, F_Y)$) given $X$:

$$\nu(F_Y) = \int E[RIF(y; \nu, F_Y)|X = x].dF_X(x)$$

where $RIF(y; q_\tau) = q_\tau + \frac{\tau - \frac{1}{T}(y \leq q_\tau)}{f_y(q_\tau)}$ and $q_\tau$ is the $\tau$th sample quantile. The above results show that when we are interested in the impact of covariates on a specific distributional statistic $\nu(F_Y)$ such as quantile, we simply need to integrate over $E[RIF(y; \nu, F_Y)|X = x]$.

### 2.1 Identification

Imbens and Newey (2009) develop the control variable approach to identify and estimate models with nonseparable, multidimensional disturbances with the presence of endogeneity. The main advantage of control variable is that by conditioning on this variable, regressors and disturbances become independent. Rothe (2010) show that in the triangular model, unconditional partial effects can be identified under certain regularity conditions as shown by Imbens and Newey (2009). The below result shows that Unconditional Quantile Partial Effects can be identified by using the control variable.
Proposition 1: In the triangular model of equation (1) and (2), under the conditions shown by Imbens and Newey (2009) in Theorem 1, we have control variable \( V = F_X|Z \) such that \( \epsilon \perp X|V \) then the unconditional quantile partial effect of \( X \) on \( F_Y(\tau) \), is given by 
\[
\alpha(F_Y) = -E[\delta_X(\tau, X, V)]/\delta q_\tau(F_Y(q_\tau, X, V)).
\]

The above result is a special case of theorem 1 in Rothe (2010). Rothe’s (2010) result holds for any distributional statistic whereas our results focus only in quantiles so we need fewer regularity conditions. The usefulness of this result is to identify the re-centered influence function. Rothe (2010) points out that we can find control variable \( V = F_X|Z \) in this setup because the exclusive source of dependence between \( X \) and \( \epsilon \) is their joint dependence on the disturbance term \( \eta \) from equation 2. This makes control variable \( V \) a one to one transformation of \( \eta \) and we obtain the result \( \epsilon \perp X|V \).

We can identify the unconditional quantile partial effect in a nonseparable model by showing that the average marginal effect is the expectation of the marginal change in the distribution of \( Y \) condition on both \( X \) and the control variable \( V \). This result holds because control variable \( V = F_X|Z \) is a simple transformation of the endogenous regressors while the conditional independence condition \( X \perp \epsilon|V \) holds.

In this paper, we focus on estimating the RIF regression for quantiles. In many cases, researchers are interested in finding out the change in the distributional statistic over a time period or across a different group. For example, in wage decomposition literature, we are interested in finding out the change in the distributional statistics like quantile, Gini coefficient etc over two time period. Fipro et al. (2009) denote the so called RIF regression as,
\[
m = E[RIF(y, \nu)|X = x, V = v]
\]
(5)
The estimation of RIF regression is closely related to the estimation of UQPE. We explore this relation to estimate the RIF regression function in our next result as shown below.

Lemma 1: Suppose that the conditions of Proposition 1 hold and \( \alpha(Y_{\tau}) \) is continuous and at least twice differential at \( Y_{\tau} \), then we can estimate the RIF regression function by integrating over \( \alpha(Y_{\tau}) \), that is,
\[
m(\tau) = \int \alpha_{Y,X,V}(X = x, V = v, q_\tau, Y_{\tau}) dx
\]
(6)
This result shows that we can identify the RIF regression function by using the unconditional partial effect of \( X \) on the distributional statistic, such as quantile, as the building block. Rothe (2010) points out that if the Hadamard differentiability condition is satisfied, then
the smoothness property is fulfilled for moments, quantiles, and inequality measures like Gini coefficient and Lorenz curve. However, to estimate the RIF regression by using the unconditional quantile partial effect, we only need $\alpha(\tau_Y)$ to be continuous and at least twice differential. So the additional restriction needs to be satisfied to establish the general result for any distributional statistic.

This result also allows us to represent a close relation between the unconditional partial effect and RIF regression. This result is not specific for a model with endogenous regressors only; it can be applied for the case of exogenous regressors, too. The advantage of this representation is that we can easily identify the RIF regression function by exploiting the relationship between UQPE and RIF in a general setting with the control variable.

2.2 Additively Separable Model

In this section, we consider a special class of model as shown in equation (1). We assume an additively separable model; however, we do not put any restrictions on the dependency structure between $X$ and $\epsilon$. Our new model has the following form,

$$Y = \tilde{h}(X'\beta + g(X, \epsilon))$$

where $\tilde{h}$ is differentiable and strictly monotonic. The above model is a generalization of the case $Y = h_0(X'\beta + \epsilon)$ as considered by Fipro et al. (2009). The additional complexity of this model is the dependency structure between $X$ and $\epsilon$ through the function $g(\cdot)$. We assume $g(\cdot)$ is continuous and twice differentiable.

Fipro et al. (2009) show that in a linear additive model the Unconditional Partial Effect (UQPE) of $X$ on the $\tau$th quantile of $Y$ is $UQPE(\tau) = \beta \times h_0'(h_0^{-1}(q_{\tau}))$ by using the independence assumption between regressors and errors. In this paper, we estimate the UQPE for quantiles in a general setting by using the control variable technique developed by Imbens and Newey (2009).

**Proposition 2:** Assuming that the structural form $Y = \tilde{h}(X'\beta + g(X, \epsilon))$ is strictly monotonic in $\epsilon$ and $\tilde{h}(\cdot)$ and $g(\cdot)$ and both are continuous and twice differentiable then we have,

$$UQPE(\tau) = \beta \times g'(X, V, \tilde{h}'(\tilde{h}^{-1}(q_{\tau})))$$

This result is a generalization of proposition 1 in Fipro et al. (2009) in the presence of endogenous regressors. Fipro et al.’s (2009) results can be obtain from the above proposition. Assume that $g(X, \epsilon) = \epsilon$ then $g'(\cdot) = (0, 1)'$ and we go back to the result $UQPE(\tau) =$
\[ \beta \times \tilde{h}' \left( \tilde{h}^{-1}(q_r) \right) \] as shown by Fipro et al. (2009). The effect of \( X_j \) on the \( \tau \)th quantile is the index \( \beta_j \) times the slope of the transformation function evaluated at this point \( g' \left( X, V, \tilde{h}'(\tilde{h}^{-1}(q_r)) \right) \).

As we mentioned earlier, the main focus of this paper is to estimate the RIF regression function \( \tilde{m} \), so we use the above result to identify \( \tilde{m} \) in a nonlinear additive model. By following the steps of the proof of Lemma 1, we can show that \( \tilde{m} \) in a nonlinear additive model is given by:

\[ m = \beta \int g' \left( X = x, V = v, \tilde{h}'(\tilde{h}^{-1}(q_r)) \right) dx \tag{9} \]

The above result is a straight application of Lemma 1. By using the relation between UQPE and RIF regression function, we can derive the result from equation 5.

### 2.3 Box-Cox Model

We want to illustrate our theoretical model by showing an example. We consider the famous Box-Cox transformation of equation 1, so we can rewrite equation 1 as,

\[ Y = h^*(X' \beta_r, \lambda_r, g(X, \epsilon)) \tag{10} \]

where \( h^* \) is strictly monotonically increasing in \( X' \beta_r \) and also depends on the shape parameter \( \lambda_r \). Moreover, \( Y > 0, X \in \mathbb{R}^k \) are observed, while the parameter \( \beta_r \in \mathbb{B} \) and \( \lambda_r \in \mathbb{R} \) are unknown and \( \tau \in (0, 1) \). So \( \tilde{h} \) is given by

\[ h^* = \begin{cases} (X' \beta_r + g(X, \epsilon))^\lambda_r - 1) / \lambda_r & \text{if } \lambda_r \neq 0 \\ \log(X' \beta_r + g(X, \epsilon)) & \text{if } \lambda_r = 0 \end{cases} \tag{11} \]

assuming \( \lambda_r \in [\underline{\lambda}_r, \bar{\lambda}_r] \) to be a finite closed interval. The advantage of using the Box-Cox transformation is that the shape parameter \( \lambda \) adjusts with each quantile to capture the precise shape of the conditional expectation function. Using the above result, one can obtain that under this specification, the UQPE is given by,

\[ UQPE(\tau) = \beta_r \times g' \left( X, V, (1 + \lambda_r q_r)^{(\lambda_r - 1)/\lambda_r} \right) \tag{12} \]

and thus the RIF regression of \( X \) on the \( \tau \)th quantile of \( Y \) is,

\[ m^*(\tau) = \int \beta_r \times g' \left( X, V, (1 + \lambda_r q_r)^{(\lambda_r - 1)/\lambda_r} \right) dx \tag{13} \]
The results in equation (9) and (10) come from the results in equation (5) and (6) respectively by showing that \( h^{*'}(h^{*-1}(q_r)) = (1 + \lambda r q_r)^{(\lambda r - 1)/\lambda r} \). Now we do an experiment by completely ignoring the dependency structure between the regressors and the error terms. In practical application, in many cases, researchers may not be able to follow the control variable technique because they may not have valid instruments. The goal of this experiment is to find out the range of the bias term for different functional forms if we do not follow the control variable approach while there are some endogenous regressors present in X.

We consider the same Box-Cox model as shown in equation (6) and to simplify notation we assume that there is only one endogenous regressor and no exogenous regressor present in \( X \). We need to assume some functional form of \( g \), so we take a general multiplicative form which is given by,

\[
g(X, \epsilon) = \theta(x) \times \epsilon \tag{14}
\]

where \( \theta(x) \) is independent of \( \epsilon \) and we don’t put any restrictions on the functional form of \( \theta \). By using this simple setup, we can show that the UQPE is,

\[
UQPE(\tau) = \beta_r (1 + \lambda r q_r)^{(\lambda r - 1)/\lambda r} + (1 + \lambda r q_r)^{(\lambda r - 1)/\lambda r} \left( \frac{\int \theta'(X) \left((1 + \lambda q_r)^{1/\lambda} - X' \beta_r\right) f_\epsilon(\cdot) dF_X(x)}{\int \theta(\cdot) f_\epsilon(\cdot) dF_X(x)} \right) \tag{15}
\]

Suppose we ignore the dependence structure between \( X \) and \( \epsilon \). This means \( g(X, \epsilon) = \epsilon \) where \( \theta = 1 \). By plugging the value \( \theta' = 0 \) we go back to the result,

\[
UQPE(\tau) = \beta_r (1 + \lambda r q_r)^{(\lambda r - 1)/\lambda r} \tag{16}
\]

which can be derived from the result by Fipro and et al. (2009). So we have the bias term,

\[
B_{UQPE} = (1 + \lambda r q_r)^{(\lambda r - 1)/\lambda r} \left( \frac{\int \theta'(X) \left((1 + \lambda q_r)^{1/\lambda} - X' \beta_r\right) f_\epsilon(\cdot) dF_X(x)}{\int \theta(\cdot) f_\epsilon(\cdot) dF_X(x)} \right) \tag{17}
\]

We note that we can separate out the bias term from the parameter estimate because of our assumption of a linearly additive model. Once this assumption violates we cannot separate out the bias term additively as shown in the above equation. Again for a linear additive model, we set \( \lambda_r = 1 \) and get \( UQPE(\tau) = \beta_r \) as Fipro et al. (2009) has shown for a linear case. The RIF regression of \( X \) on the \( \tau \)th quantile of \( Y \) is,

\[\text{First we note that } h^{*-1}(q_r) = (1 + \lambda r q_r)^{1/\lambda r}, \text{ then show } h^{*'}(h^{*-1}(q_r)) = 1/(1 + \lambda r q_r)^{(1-\lambda_r)/\lambda r}.\]
\[ m(\tau) = (1 + \lambda \tau q_\tau)^{(\lambda \tau - 1)/\lambda \tau} X'\beta + \]
\[ (1 + \lambda \tau q_\tau)^{(\lambda \tau - 1)/\lambda \tau} \left( \frac{\int \theta'(X) \left( (1 + \lambda \tau q_\tau)^{1/\lambda} - X'\beta_\tau \right) f_\epsilon(\cdot)dF_X(x)}{\int \theta(X) f_\epsilon(\cdot)dF_X(x)} \right) dX \quad (18) \]

Similarly the bias term for the RIF regression is given by,

\[ B_{RIF} = (1 + \lambda \tau q_\tau)^{(\lambda \tau - 1)/\lambda \tau} \left( \frac{\int \theta'(X) \left( (1 + \lambda \tau q_\tau)^{1/\lambda} - X'\beta_\tau \right) f_\epsilon(\cdot)dF_X(x)}{\int \theta(X) f_\epsilon(\cdot)dF_X(x)} \right) dX \quad (19) \]

Figure 1: Markup Factor

We note that the bias terms for both UQPE and RIF regression depends on the markup factor \((1 + \lambda q)^{(\lambda - 1)/\lambda}\). The function is not monotonically increasing in certain ranges of \(\lambda\) and \(q\) as we have shown in figure 1. In both the panels of figure 1, we show the 3D plot of the markup factor where the first two axes are \(\lambda\) and \(q\). In the left panel, we plot the values for \(\lambda\) and \(q\) from 0 to 5, whereas in the right panel we show the the values for \(\lambda\) and \(q\) from 0 to 1. From the left panel, we see that the function is monotonically increasing for higher values of \(\lambda\) and \(q\) and from the right panel we see this property does not hold for smaller ranges of values of \(\lambda\) and \(q\).

The bias terms also depends on the factor \(\frac{\theta'(X)}{\theta'(X)}\). We can easily show that if \(\theta(X)\) is a monotonically increasing function of \(X\), then \(\frac{\theta'(X)}{\theta'(X)}\) is a monotonically decreasing function.
of $X$. Since the markup factor is non-monotonic and $\frac{\theta'(X)}{\theta(X)}$ is a monotonically decreasing function, we can conclude that the bias terms are non-monotonic across the quantiles. However, if we consider two different functions $\theta_1(X)$ and $\theta_2(X)$ such that $\theta_1(X) > \theta_2(X)$ for all $X$, this implies that the correlation between $X$ and $\epsilon$ is higher for $\theta_1(X)$ compared to $\theta_2(X)$. Then the bias terms for $\theta_1(X)$ are higher compared to $\theta_2(X)$ for both UQPE and RIF regression function.

3 Conclusion

In this paper, we identified the UQPE and RIF regression function introduced by Fipro et al. (2009) in a non-separable triangular model with endogenous regressors. We follow the same setup of the model introduced by Imbens and Newey (2009). Rothe (2010) shows the identification of unconditional partial effects in the same setup. Our first result of the identification of UQPE is a simpler version of Rothe’s (2010) result because we consider only quantile whereas Rothe’s (2010) result holds for any distributional statistic. However, we use this result to identify the unconditional quantile regression function in the presence of the endogenous regressors, which is the main focus of this paper.

We have also considered a nonlinear additively separable model where the additive separability condition between $X$ and $\epsilon$ is a special case of the nonseparable model. We focus on this type of model because a linear additively separable model is the most commonly used model in the micro economic applications, and the results for linear model is a special case of our nonlinear model. We have identified the UQPE and RIF regression function for the nonlinear additively separable model by using the control variable approach and show that Fipro et al.’s (2009) results are a special case of the result by assuming no endogenous regressors in $X$. All these results are not specific for models with endogenous regressors, but they apply to the exogenous case where there is no need for control variable.

Linear and log-linear models are special cases of the Box-Cox model, so the results for linear and log-linear models can be obtained from our results. We can separate out the bias terms for UQPE and RIF regression once we assume the error term has multiplicative form between $X$ and $\epsilon$. This can be done only in the additively separable model. We show that the bias terms for UQPE and RIF regression function are non-monotonic across quantiles, however, the bias terms have higher values when the error term is strongly correlated with endogenous regressors.
Appendix

Proof of Proposition 1

In a nonseparable model we have the structural form \( Y = h(X, \epsilon) \), where \( h(\cdot) \) is assumed to be strictly monotonic in \( \epsilon \). We also assume that \( h(\cdot) \) follows a distribution \( F_{h(\cdot)} \). The resulting probability response model is,

\[
Pr[Y > q_r | X = x, V = v] = Pr[h(X, \epsilon) > q_r | X = x, V = v] = 1 - Pr[h(X, \epsilon) \leq q_r | X = x, V = v] = 1 - F_{h(\cdot)}(q_r, X, V)dF_{X|V}
\]

We first take the derivative with respect to \( X_j \) and then integrate over the distribution of \( X \) to obtain the marginal effects,

\[
\frac{\delta Pr[Y > q_r | X = x, V = v]}{\delta x_j} = \int \frac{d}{dx_j} [1 - F_{h(\cdot)}(q_r, X, V)] dF_{X|V} = \frac{\delta}{\delta x_j} \int [1 - F_{h(\cdot)}(q_r, X, V)] dF_{X|V} = -\frac{\delta}{\delta x_j} [E(F_{h(\cdot)}(q_r, X, V))] = -E \left[ \frac{\delta}{\delta x_j} (F_{h(\cdot)}(q_r, X, V)) \right] = -E \left[ \delta x_j (F_{h(\cdot)}(q_r, X, V)) \right]
\]

We note that once we condition the covariates \( X \) on the control variable \( V \), the error term \( \epsilon \) in \( h(\cdot) \) become independent of \( X \) and we can separate out that two parts. We can rewrite \( f_Y(q_r) \) as

\[
f_Y(q_r) = \frac{\delta}{\delta q_r} (F_Y(q_r)) = \frac{\delta}{\delta q_r} [Pr(g(y, X, V) < q_r)|X = x, V = v] = \frac{\delta}{\delta q_r} [F_Y(q_r, X, V)] = \delta q_r (F_Y(y, X, V))
\]
Fipro, Fortin and Lemieux (2009) show that ‘Unconditional Quantile Partial Effects’ can be represented as,

\[
\alpha(F_Y) = \int \frac{\delta}{\delta x_j} (Pr[Y > q_r | X = x, V = v]) dF_{X|V} f_Y(q_r)
\]

We substitute the results of marginal effects and \(f_Y(q_r)\) to the above equation to obtain the UQPE in nonseparable model,

\[
\alpha(F_Y) = -\frac{E[\delta_X (F_Y(q_r, X, V))]}{\delta q_r (F_Y(q_r, X, V))}
\]

**Proof of Lemma 1**

Again Fipro, Fortin and Lemieux (2009) show that there exists a close relation between the unconditional partial effects (\(\alpha_{F_Y}\)) and RIF regression function \(m_r\). This can be represented as,

\[
E\left[\frac{dE[RIF(y, q_r)]}{dx}|X = x, V = v]\right] = \alpha_{F_Y}(X = x, V = v, q_r)
\]

\[\Rightarrow \int \frac{dE[RIF(y, q_r)]}{dx}|X = x, V = v| dF_{X|V} = \alpha_{F_Y}(X = x, V = v, q_r)
\]

we substitute RIF regression function \(m_r = E[RIF(y, q_r)]|X = x, V = v]\) in the above relation and get,

\[
\int \frac{dm_r}{dx} dF_{X|V} = \alpha_{F_Y}(X = x, V = v, q_r)
\]

\[\Rightarrow \int dm_r \int dF_{X|V} = \int \alpha_{F_Y}(X = x, V = v, q_r) dx \]

\[\Rightarrow m_r = \int \alpha_{F_Y}(X = x, V = v, q_r) dx \]
Proof of Proposition 2

In additive separable model, we have $Y = \tilde{h}(X'\beta + g(X, \epsilon))$ where $g(X, \epsilon)$ is assumed to be continuous and twice differential in it’s both argument. We also assume that $g(\cdot)$ follows a distribution $F_g(\cdot)$. The resulting probability model is,

$$Pr[Y > q_r | X = x, V = v] = Pr[\tilde{h}(X'\beta + g(X, \epsilon)) > q_r | X = x, V = v]$$

$$= Pr\left[ g(X, \epsilon) > \tilde{h}^{-1}(q_r) - X'\beta | X = x, V = v \right]$$

$$= 1 - Pr\left[ g(X, \epsilon) \leq \tilde{h}^{-1}(q_r) - X'\beta | X = x, V = v \right]$$

$$= 1 - F_g(\tilde{h}^{-1}(q_r) - X'\beta)$$

To obtain the marginal effects we first take the derivative with respect to $X_j$ and then integrate over the distribution of $X$,

$$\int \frac{\delta Pr[Y > q_r | X = x, V = v]}{\delta x_j} dF_{X|V} = \int \frac{\delta}{\delta x_j} (1 - Pr[Y \leq q_r | X = x, V = v]) dF_{X|V}$$

$$= \frac{\delta}{\delta x_j} \int \left[ 1 - F_g(\tilde{h}^{-1}(q_r) - X'\beta) \right] dF_{X|V}$$

$$= - \frac{\delta}{\delta x_j} \left[ E\left(F_g((\tilde{h}^{-1}(q_r) - X'\beta))\right) \right]$$

$$= \beta_j E\left[f_g(\tilde{h}^{-1}(q_r) - X'\beta)\right]$$

where $f_g(\cdot)$ is the marginal density of $g(\cdot)$. We note that $F_g(\cdot)$ depends on both $g(\cdot)$ and $(q_r - X'\beta)$. That’s why we get marginal density function as $f_g(\cdot)\left(g'(\tilde{h}^{-1}(q_r) - X'\beta)\right)$.

We can rewrite $f_Y(q_r)$ as,
\[ f_Y(q_r) = \frac{\delta}{\delta q_r} (F_Y(q_r)) \]
\[ = \frac{\delta}{\delta q_r} E \left[ Pr \left( \tilde{h}(y, X, V) < q_r \right) | X \right] \]
\[ = \frac{\delta}{\delta q_r} E \left[ g(X, \epsilon) < \tilde{h}^{-1}(q_r) - X' \beta | X \right] \]
\[ = E \left[ \frac{\delta}{\delta q_r} \left[ F_{g(\cdot)} \left( \tilde{h}^{-1}(q_r) - X' \beta \right) \right] \right] \]
\[ = E \left[ \frac{f_{g(\cdot)} \left( g'(\tilde{h}^{-1}(q_r) - X' \beta) \right)}{g'(\tilde{h}'(\tilde{h}^{-1}(q_r)))} \right] \]

To obtain the Unconditional Partial Effect, we substitute the results of marginal effects and \( f_Y(q_r) \) into Fipro et al. (2009) result,

\[ UQPE(\tau) = \int \frac{\delta}{\delta X_j} \left( Pr[Y > q_r | X = x, V = v] \right) dF_X|V \]
\[ f_Y, \delta(q_r) \]
\[ = \beta_j E \left[ f_{g(\cdot)} \left( g'(\tilde{h}^{-1}(q_r) - X' \beta) \right) \right] / E \left[ \frac{f_{g(\cdot)} \left( g'(X, \epsilon) \leq \tilde{h}^{-1}(q_r) - X' \beta \right)}{g'(\tilde{h}'(\tilde{h}^{-1}(q_r)))} \right] \]
\[ = \beta_j g' \left( \tilde{h}'(\tilde{h}^{-1}(q_r)) \right) \]

**Derivation of Unconditional Quantile Partial Effects in Box-Cox Model**

In the Box-Cox model, a small change in \( t \) in a covariate \( X_j \) corresponds not only a simple location shift of the distribution \( Y \), but also it captures of the effect on the other distributional features. For the sake of simplicity, assume that \( \epsilon \) follows a distribution \( F_\epsilon \). Then the resulting probability response model is,
\[
Pr[Y_\lambda > q_\tau | X = x] = Pr\left[\frac{(X'\beta_\tau + g(X, \epsilon))^{\lambda} - 1}{\lambda} > q_\tau | X = x\right]
= Pr\left[g(X, \epsilon)) > (1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau | X = x\right]
= Pr\left[\theta(X)\epsilon > (1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau | X = x\right]
= Pr\left[\epsilon > \frac{(1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau}{\theta(X)} | X = x\right]
= 1 - F_\epsilon\left[\frac{(1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau}{\theta(X)} | X = x\right]
\]

Thus if \( \epsilon \) was normally distributed, the probability response model would be a standard probit model. Taking derivative with respect to \( X \) yields,

\[
\frac{dPr[Y_\lambda > q_\tau | X = x]}{dX} = \left(\frac{\beta}{\theta(X)} + \frac{\theta'(X)}{\theta^2(X)} \left((1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau\right)\right) \times f_\epsilon\left[\frac{(1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau}{\theta(X)}\right]
\]

where \( f_\epsilon(\cdot) \) as the density of \( \epsilon \) and the marginal effects are obtained by integrating over the distribution of \( X \). The average marginal effect is

\[
\int \frac{dPr[Y_\lambda > q_\tau | X = x]}{dX}dF_X(x) = \int \left(\frac{\beta}{\theta(X)} + \frac{\theta'(X)}{\theta^2(X)} \left((1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau\right)\right) \times f_\epsilon\left[\frac{(1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau}{\theta(X)}\right]dF_X(x)
\]

We can rewrite \( f_Y(q_\tau) \) as,
\[ f_Y(q_\tau) = \frac{d}{dq_\tau}(F_Y(q_\tau)) \]
\[ = \frac{d}{dq_\tau}[Y \leq q_\tau(x)] \]
\[ = \frac{d}{dq_\tau} \left[ Pr(X'\beta_\tau + g(X, \epsilon))^{\lambda_\tau} - 1 \leq q \mid X = x \right] \]
\[ = \frac{d}{dq_\tau} \left[ Pr(g(X, \epsilon) \leq (1 + \lambda q_\tau)^{1/\lambda_\tau} - X'\beta_\tau \mid X = x) \right] \]
\[ = \frac{d}{dq_\tau} \left[ Pr(\theta(X) \epsilon \leq (1 + \lambda q_\tau)^{1/\lambda_\tau} - X'\beta_\tau \mid X = x) \right] \]
\[ = \frac{d}{dq_\tau} \left[ Pr(\epsilon \leq \left( \frac{(1 + \lambda q_\tau)^{1/\lambda_\tau} - X'\beta_\tau}{\theta(X)} \right) \mid X = x) \right] \]
\[ = \frac{d}{dq_\tau} \left[ F_\epsilon \left( \frac{(1 + \lambda q_\tau)^{1/\lambda_\tau} - X'\beta_\tau}{\theta(X)} \right) \right] \]
\[ = E \left[ \frac{f_\epsilon(\cdot)}{\theta(X)} \times \frac{1}{\lambda_\tau} (1 + \lambda q_\tau)^{(1/\lambda_\tau) - 1} \times \lambda_\tau \right] \]
\[ = \left( (1 + \lambda q_\tau)^{(1 - \lambda_\tau)/\lambda_\tau} \right) E \left[ \frac{1}{\theta(X)} f_\epsilon \left( \frac{((1 + \lambda q_\tau)^{1/\lambda_\tau} - X'\beta_\tau)}{\theta(X)} \right) \right] \]
\[ = \left( (1 + \lambda q_\tau)^{(1 - \lambda_\tau)/\lambda_\tau} \right) \int \frac{1}{\theta(X)} f_\epsilon \left( \frac{((1 + \lambda q_\tau)^{1/\lambda_\tau} - X'\beta_\tau)}{\theta(X)} \right) dF_X(x) \]

The Unconditional Quantile Partial Effect for the Box-Cox model is given by,

\[ UQPE = \frac{\int \left( \frac{\beta_\tau}{\theta(x)} + \frac{\theta(x)}{\theta^2(x)} \left( (1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau \right) f_\epsilon(\cdot) \right) dF_X(x)}{((1 + \lambda q_\tau)^{(1 - 1)/\lambda}) \int \frac{1}{\theta(x)} f_\epsilon(\cdot) dF_X(x)} \]
\[ = (1 + \lambda q_\tau)^{(\lambda - 1)/\lambda_\tau} \left( \beta_\tau + \int \frac{\theta(x)}{\theta^2(x)} \left( (1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau \right) f_\epsilon(\cdot) dF_X(x) \right) \frac{1}{\theta(x)} f_\epsilon(\cdot) dF_X(x) \]
\[ = \beta_\tau (1 + \lambda q_\tau)^{(\lambda - 1)/\lambda_\tau} + (1 + \lambda q_\tau)^{(\lambda - 1)/\lambda_\tau} \left( \int \frac{\theta(x)}{\theta^2(x)} \left( (1 + \lambda q_\tau)^{1/\lambda} - X'\beta_\tau \right) f_\epsilon(\cdot) dF_X(x) \right) \frac{1}{\theta(x)} f_\epsilon(\cdot) dF_X(x) \]

By using the result of Lemma 2, the RIF regression function \( m_\tau \) can be written as,
\begin{align*}
m_{\tau} &= \int (1 + \lambda_{\tau} q_{\tau})^{(\lambda_{\tau}-1)/\lambda_{\tau}} \left( \beta_{\tau} + \int \frac{\theta'(X)}{\theta(X)} \left( (1 + \lambda q_{\tau})^{1/\lambda} - X' \beta_{\tau} \right) f_\epsilon(\cdot) dF_X(x) \right) dX \\
&= (1 + \lambda_{\tau} q_{\tau})^{(\lambda_{\tau}-1)/\lambda_{\tau}} X' \beta + (1 + \lambda_{\tau} q_{\tau})^{(\lambda_{\tau}-1)/\lambda_{\tau}} \int \left( \int \frac{\theta'(X)}{\theta(X)} \left( (1 + \lambda q_{\tau})^{1/\lambda} - X' \beta_{\tau} \right) f_\epsilon(\cdot) dF_X(x) \right) dX
\end{align*}

References


—— Supplement to ‘Unconditional Quantile Regressions’. Econometrica supplemental material 77.


